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Are the critical exponents for Anderson localization due to disorder well understood?

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Abstract Semiclassical theory has been used, with some success, to discuss Anderson localization due to disorder. Our attention is focused on the quantum–chemical network model via a Boltzmann–like equation, and García-García's semiclassical approach, contacted with early work of Care and March on compensated semiconductor. This work is related with the recent semiclassical treatment on the effect of disorder on the nature of electron states in the quantum–chemical network model.

Keywords Anderson localization · Semiclassical theory · Critical exponent · Disorder · Dimensionality · Scaling

It is well known that the phenomenon of localization depends on the dimensionality d. However, the only mathematically rigorous statement concerning the existence of extended states in one dimension is that for any finite amount of disorder there are no extended states in $d = 1$ [1]. The general belief, based on the one-parameter scaling method (an approximate theory), is that there are also no extended states in

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two dimensions, even for infinitesimally small disorder. This question cannot be considered to be decisively solved. But in what follows, it is relevant to emphasize the early study of Dancz, Edwards and March [2], using what is termed the disordered quantum network model in two dimensions. These authors employed a semiclassical approach, which led them to a Boltzmann–like equation [2,3]. March and Angilella [3] discussed a two-dimensional square lattice with a band potential which permits analytic solution, gave examples of the quantum network energy band dispersion relations in a graphene layer or a related B layer or B nanotubes of different chiralities, and the corresponding density of states. Some attention was then given to the effect of disorder of a quantum network model [3]. Using a tree assumption (no closed circuits), a Boltzmann equation was derived for the logarithmic derivative of the density of states [3]. Their conclusion for the above model was that indeed all the states were localized in two dimensions.

Here, therefore, we shall make this assumption for Anderson localization as a starting point. But then, the focus of this paper will be concerned with the critical exponents s and ν, to be defined immediately below. A relationship between the conductivity exponent s in $\sigma \sim |\tau - \tau*|^{s}$ and the localization length exponent ν in $\lambda \sim |\tau - \tau*|^{-\nu}$ is [4]:

$$s = (d - 2)\,\nu\,. \tag{1}$$

The numerical value for critical exponent ν was evaluated by using the $\in$ expansion [5] or diagrammatic techniques [6], $s = \nu = 1$. However, it is understood that the $\in$ expansion cannot be exact for $\in = 1$.

The important early work of Wegner [7,8] recognized that with $\in = d - 2$, $\in$ is not a small parameter any more for d = 3 systems which are paradigmatic for a metal – insulator (MI) transition. This situation is somewhat analogous to that argued by one of us [9] for the 3d Ising model, for which the validity of the $\in$ expansion (now with $\in = 4 - d$) was also brought into question. Here we would like to mention briefly some recent progresses on the 3d Ising model [10–12]. Lawrynowicz et al. [11] reformulated the algebraic part of the theory in terms of the quaternionic sequence of Jordan algebras to look at some of the geometrical aspects of simple orthorhombic Ising lattices, and represented a mathematical outlook of fractals and chaos related to these 3D Ising lattices [12]. However, it should be noticed that the 3d Ising model has been used to describe the critical behaviours for magnetic materials like insulating ferromagnet $CrBr_3$ [13], not for Anderson localization.

Returning to the one–parameter scaling hypothesis, already mentioned above [14–18], it seems fair to say that the critical exponents thereby obtained were only partially in agreement with the results of other theories and also with experimental data.

It is of some important at this point to refer to the study of Chayes et al. [19]. These authors derived, for a system with statistically uncorrelated randomness, the following inequality

$$\nu > 2/d, \tag{2}$$

with the assumption of the validity of a one – parameter scaling law. In the same context of Eq. (2), Harris [20] concluded that the inequality (2) was satisfied in order for fluctuations to be irrelevant.

Furthermore, one can summarize results available from numerical scaling calculations as follows. Within the statistical accuracy of the raw data ($\sim$1%), in almost all cases the one–parameter scaling form can be established when one is sufficiently close to the transition. There is a quantitative criterion for the validity of the scaling behaviour. The method yields complete localization in two dimensions with an essential singularity at zero disorder and an Anderson transition [21] in three dimensions.

Theoretically, the critical exponent ν is expected to be independent of which side of the fixed point it is derived from. Experimentally, on doped semiconductors, the critical exponents studied by measuring the dielectric susceptibility and the conductivity turned out, indeed, to be equal on both the insulating and metallic sides of the Anderson transition, respectively. However, the experiments, to which we shall return briefly below, yielded a variety of critical exponents. These for Si:P, studied by Paalanen and Thomas [22], and also for other uncompensated doped materials, $s = \nu = 1/2$. But for $Al_xGa_{1-x}As$ [23], other amorphous materials and compensated semiconductors, $s = \nu = 1$. The critical exponent ν in Ge:Sb changes between 1/2 and 1 depending on compensation [24].

We turn next to more recent computational and theoretical studies. In particular, Markoš [25] reported numerical studies on the metal–insulator transition in the 3d Anderson model, based on the finite–size scaling analysis of the smallest positive Lyapunov exponent. Markoš also verifies numerically that the same scaling holds also for higher Lyapunov exponents, and stresses that this lends support to the one–parameter scaling theory of localization discussed above. This same study gives the critical exponent ν in the range $1.50 \leq \nu \leq 1.54$, in very good agreement with MacKinnon [26]: see also [14,27].

At the outset, we referred to the 2d theoretical study by Dancz et al. [2] of the disordered quantum network model: see also [3]. By using semiclassical ideas, these workers passed to a Boltzmann – like equation treatment. The above results for ν given by Markoš [25] prompt us then to emphasize the different semiclassical theory used, but now in d dimensions, by García-García [28]. The main results of this study is the proposal that [28]

$$\nu = \frac{1}{2} + \frac{1}{d-2} \tag{3}$$

which embraces the range obtained numerically by Markoš [25], namely $1.50 \leq \nu \leq 1.54$, since Eq. (3) yields $\nu = 3/2$ for $d = 3$. It is also relevant to refer here to the work of Wang and García-García [29] who obtain a critical exponent $\nu = 1.61 \pm 0.25$ for Anderson localization in a 3d kicked rotor.

Before returning to the experiments on compensated semiconductors, it is of obvious interest to refer to other work on scaling. Based on spectral statistics obtained in numerical simulations on 3d disordered systems within the tight–binding approximation, Varga et al. [30] have proposed a superuniversal scaling relation which allows

one to collapse data for the orthogonal, unitary, and symplectic symmetries onto a single scaling curve.

A remaining question we wish to address next concerns the spread in experimentally extracted critical values ν for compensated semiconductors. In this context, we draw attention to the early studies of Care and March [31,32]. These workers stressed that in compensated semiconductors, one must expect a 'competition' between Anderson and Wigner – type localization, the latter, of course being driven by Coulombic repulsions e^2/r_{ij} between electrons i and j at separation r_{ij} [32]. Though the work in [31] and [32] obviously needs refinement as it was completed some four decades ago, the experimental variations of extracted ν values are consistent, at least qualitatively, with such a competition between Anderson and Wigner localization, as the degree of compensation is varied. In addition, therefore, to a systematic quantitative study of the effects of Coulomb repulsion on the critical exponent ν, we stress two further areas of interest for future. These are: (a) a rigorous proof for the statement (see also Eq. (3) above due to García-García [28]) that the upper critical dimension is infinity, and (b) a careful study of the dimensionality dependence of critical exponents for different universality classes. The systematic studies on the critical exponents of Anderson localization would benefit to have a deeper understanding not only on the nature of the metal – insulator transition, but also on the behaviors of waves, including classic waves, light, microwaves, acoustics, etc., in complex materials [33].

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